A Functorial Approach to the Infinitesimal Theory of Groupoids

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Abstract

Lie algebroids are by no means natural as an infinitesimal counterpart of groupoids. In this paper we propose a functorial construction called *Nishimura algebroids* for an infinitesimal counterpart of groupoids. Nishimura algebroids, intended for differential geometry, are of the same vein as Lawvere's functorial notion of *algebraic theory* and Ehresmann's functorial notion of theory called *sketches*. We study *totally intransitive* Nishimura algebroids in detail. Finally we show that Nishimura algebroids naturally give rise to Lie algebroids.

1 Introduction

Many mathematicians innocently believe that *infinitesimalization* is no other than *linearization*. We contend that linearization is only a tiny portion of infinitesimalization. It is true that Lie algebras are the linearization of Lie groups, but it is by no means true that Lie algebras are the infinitesimalization of Lie groups. The fortunate success of the theory of Lie algebras together with their correspondence with Lie groups unfortunately enhanced their wrong conviction and blurred what are to be really the infinitesimalization of groups and, more generally, groupoids.

In this paper we propose, after the manners of Lawvere's functorial construction of *algebraic theory* and Ehresmann's functorial notion of theory called *sketches,* a functorial construction of *Nishimura algebroids* for the infinitesimalization of groupoids. After giving some preliminaries and fixing notation in the coming section, we will introduce our main notion of Nishimura algebroid in 6 steps. Then we will study totally intransitve Nishimura algebroids, in which the main result is that the linear part of any totally intransitve Nishimura algebroid is a Lie algebra bundle. As our final investigation we will show that Nishimura algebroids naturally give rise to Lie algebroids.

2 Preliminaries

2.1 Synthetic Differential Geometry

Our standard reference on synthetic differential geometry is Lavendhomme [5]. In synthetic differential geometry we generally work within a good topos. If the reader is willing to know how to get such a topos, he or she is referred to Kock [4] or Moerdijk and Reyes [9]. We denote by $\mathbb{R}$ the internal set of real numbers, which is endowed with a cornucopia of nilpotent infinitesimals persuant to the general Kock-Lawvere axiom. The internal category **Inf** of *infinitesimal spaces* comes contravariantly from the external category of Weil algebras over the set of real numbers by taking $\mathrm{Spec}_{\mathbb{R}}$. In particular, the infinitesimal space corresponding to the set of real numbers as a Weil algebra is denoted by 1.We should note that every infinitesimal space $\mathcal{D}$ has a distinguished point, namely, $0_{\mathcal{D}}$ (often written simply 0), and every morphism in **Inf** preserves distinguished points. An arbitrarily chosen microlinear space M shall be fixed throughout the rest of this paper.

2.2 Groupoids

Our standard reference on groupoids is [7]. Let $\mathcal{D}$ be an object in **Inf**. Given $m \in M$ and a groupoid G over M with its object inclusion map $\mathrm{id} : M \to G$ and its source and target projections $\alpha, \beta : G \to M$, we denote by $\mathcal{A}_m^{\mathcal{D}} G$ the totality of mappings $\gamma : \mathcal{D} \to G$ with $\gamma(0_{\mathcal{D}}) = \mathrm{id}_m$ and $(\alpha \circ \gamma)(d) = m$ for any $d \in \mathcal{D}$. We denote by $\mathcal{A}^{\mathcal{D}} G$ the set-theoretic union of $\mathcal{A}_m^{\mathcal{D}} G$'s for all $m \in M$. The canonical projection $\pi : \mathcal{A}^{\mathcal{D}} G \to M$ is defined as is expected. The *anchor* $\mathbf{a}_G^{\mathcal{D}} : \mathcal{A}^{\mathcal{D}} G \to M^{\mathcal{D}}$ is defined to be simply

$$\mathbf{a}_G^{\mathcal{D}}(\gamma) = \beta \circ \gamma$$

for any $\gamma \in \mathcal{A}^{\mathcal{D}} G$, where $M^{\mathcal{D}}$ is the space of mappings of $\mathcal{D}$ into M. We note that if the groupoid G is the pair groupoid $M \times M$, then $\mathcal{A}^{\mathcal{D}}(M \times M)$ can canonically be identified with $M^{\mathcal{D}}$. We write $\mathbf{I}G$ for the inner subgroupoid of G, for which the reader is referred to p.14 of [7].

2.3 Simplicial Spaces

The notion of *simplicial space* was discussed by Nishimura [10] and [12], where simplicial spaces were called *simplicial objects* in the former paper, while they were called *simplicial infinitesimal spaces* in the latter paper. *Simplicial spaces* are spaces of the form

$$D^m\{\mathcal{S}\} = \{(d_1, ..., d_m) \in D^m | d_{i_1} ... d_{i_k} = 0 \text{ for any } (i_1, ..., i_k) \in \mathcal{S}\},$$

where $\mathcal{S}$ is a finite set of sequences $(i_1, ..., i_k)$ of natural numbers with $1 \leq i_1 < ... < i_k \leq m$. By way of example, we have $D(2) = D^2\{(1,2)\}$ and $D(3) =$

$D^3\{(1,2),(1,3),(2,3)\}$. Given two simplicial spaces $D^m\{\mathcal{S}\}$ and $D^n\{\mathcal{T}\}$, we define another simplicial space $D^m\{\mathcal{S}\} \oplus D^n\{\mathcal{T}\}$ to be

$$\begin{aligned} & D^m\{\mathcal{S}\} \oplus D^n\{\mathcal{T}\} \\ & = \{(d_1,...,d_m,e_1,...,e_n) \in D^{m+n} | d_{i_1}...d_{i_k} = 0 \text{ for any } (i_1,...,i_k) \in \mathcal{S}, \\ e_{j_1}...e_{j_l} & = 0 \text{ for any } (j_1,...,j_l) \in \mathcal{T},\ d_i e_j = 0 \text{ for any } 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \end{aligned}$$

We denote by **Simp** the full subcategory of **Inf** whose objects are all simplicial spaces. Obviously the category **Simp** is closed under direct products. The category **Simp** has finite coproducts. In particular, it has the initial object **1**, which is also the terminal object.

3 Nishimura Algebroids

Let M be a microlinear space. We will introduce our main notion of *Nishimura algebroid over* M step by step, so that the text is divided into six subsections.

3.1 *Nishimura Algebroids*$_1$

Definition 1 *A Nishimura algebroid*$_1$ *over* M *is simply a contravariant functor* $\mathcal{A}$ *from the category* **Simp** *of simplicial spaces to the category* $\mathbf{MLS}_M$ *of microlinear spaces over* M *mapping finite coproducts in* **Simp** *to finite products in* $\mathbf{MLS}_M$*.*

Given a simplicial space $\mathcal{D}$ in **Simp**, we will usually write $\pi : \mathcal{A}^{\mathcal{D}} \to M$ for $\mathcal{A}(\mathcal{D})$. In particular, we will often write $\mathcal{A}^n$ in place of $\mathcal{A}^{D^n}$. We will simply write π for the projection to M in preference to such a more detailed notation as $\pi_{\mathcal{A},\mathcal{D}}$, which should not cause any possible confusion. Given $m \in M$, we write $\mathcal{A}_m^{\mathcal{D}}$ for $\{x \in \mathcal{A}^{\mathcal{D}} \mid \pi(x) = m\}$. Given a morphism $f : \mathcal{D} \to \mathcal{D}'$ in **Simp**, we will usually write $\mathcal{A}^f : \mathcal{A}^{\mathcal{D}'} \to \mathcal{A}^{\mathcal{D}}$ for $\mathcal{A}(f)$. Given $m \in M$, there is a unique element in $\mathcal{A}_m^{\mathbf{1}}$, which we denote by $\mathbf{0}_m^{\mathbf{1}}$. Given an object $\mathcal{D}$ in **Simp**, we define $\mathbf{0}_m^{\mathcal{D}} \in \mathcal{A}_m^{\mathcal{D}}$ to be

$$\mathbf{0}_m^{\mathcal{D}} = \mathcal{A}^{\mathcal{D}\to\mathbf{1}}(\mathbf{0}_m^{\mathbf{1}})$$

Example 2 *By assigning the spac* $M^{\mathcal{D}}$ *of mappings from* $\mathcal{D}$ *into* M *to each object* $\mathcal{D}$ *in* **Simp** *and assigning* $M^f : M^{\mathcal{D}'} \to M^{\mathcal{D}}$ *to each morphism* $f : \mathcal{D} \to \mathcal{D}'$ *in* **Simp***, we have a Nishimura algebroid*$_1$ *over* M *to be called the standard Nishimura algebroid*$_1$ *over* M *and to be denoted by* $\mathcal{S}_M$ *or more simply by* $\mathcal{S}$*.*

Example 3 *Let* G *be a groupoid over* M*. By assigning* $\mathcal{A}^{\mathcal{D}}G$ *to each object* $\mathcal{D}$ *in* **Simp** *and assigning* $\mathcal{A}^f G : \mathcal{A}^{\mathcal{D}'}G \to \mathcal{A}^{\mathcal{D}}G$ *to each morphism* $f : \mathcal{D} \to \mathcal{D}'$ *in* **Simp***, we have a Nishimura algebroid*$_1$ *over* M *to be denoted by* $\mathcal{A}G$*.*

Each $\sigma \in \mathfrak{S}_n$ induces a morphism $\sigma : D^n \to D^n$ in **Simp** such that

$$\sigma(d_1,...,d_n) = (d_{\sigma(1)},...,d_{\sigma(n)})$$

for any $(d_1, d_2) \in D^2$. Given $x \in \mathcal{A}^n$, we will often write ${}^\sigma x$ for $\mathcal{A}^\sigma(x)$. It is easy to see that

$$^{\tau\sigma}x = ^\tau ({}^\sigma x)$$

for any $x \in \mathcal{A}^n$ and any $\sigma, \tau \in \mathfrak{S}_n$.

Given $x \in \mathcal{A}^n$ and $a \in \mathbb{R}$, we define $a \underset{i}{\cdot} x$ $(1 \leq i \leq n)$ to be

$$a \underset{i}{\cdot} x = \mathcal{A}^{((d_1,,,,,d_n) \in D^n \mapsto (d_1,\ldots,d_{i-1},ad_i,d_{i+1},\ldots,d_n) \in D^n)}(x)$$

3.2 *Nishimura Algebroids*$_2$

Definition 4 *A Nishimura algebroid$_1$ A over M is called a Nishimura algebroid$_2$ over M if the application of $\mathcal{A}$ to any quasi-colimit diagram in* **Simp** *results in a limit diagram.*

Remark 5 *The notion of Nishimura algebroid$_2$ over M can be regarded as a partial algebrization of microlinearity.*

Example 6 *The standard Nishimura algebroid$_1$ $\mathcal{S}_M$ over M is a Nishimura algebroid$_2$ over M. This follows simply from our assumption that M is a microlinear space.*

Example 7 *Let G be a groupoid over M. Then the Nishimura algebroid$_1$ $\mathcal{A}G$ over M is a Nishimura algebroid$_2$ over M. This follows simply from our assumption that M and G are microlinear spaces.*

Let $\mathcal{A}$ be a Nishimura algebroid$_2$ over M. Let $m \in M$ with $x, y \in \mathcal{A}^1_m$. By using the quasi-colimit diagram (1) of small objects referred to in Proposition 6 (§2.2) of Lavendhomme [5], there exists a unique $z \in \mathcal{A}^{D \oplus D}$ with $\mathcal{A}^{(d \in D \longmapsto (d,0) \in D \oplus D)}(z) = x$ and $\mathcal{A}^{(d \in D \longmapsto (0,d) \in D \oplus D)}(z) = y$. We define $x + y$ to be $\mathcal{A}^{(d \in D \longmapsto (d,d) \in D \oplus D)}(z)$. Given $a \in \mathbb{R}$, we define ax to be $\mathcal{A}^{(d \in D \mapsto ad \in D)}(x) \in \mathcal{A}^1_m$. With these operations we have

Theorem 8 *Given a Nishimura algebroid$_2$ $\mathcal{A}$ over M, $\mathcal{A}^1_m$ is an $\mathbb{R}$-module for any $m \in M$.*

Proof. The proof is essentially a familiar proof that $\mathcal{S}^1_m$ is an $\mathbb{R}$-module, for which the reader is referred, e.g., to Lavendhomme [5], §3.1, Proposition 1. What we should do is only to reformulate the familiar proof genuinely in terms of diagrams. The details can safely be left to the reader. ■

Let $\mathcal{A}$ be a Nishimura algebroid$_2$ over M with $m \in M$.

Let $x, y \in \mathcal{A}^2_m$ with

$$\begin{aligned} &\mathcal{A}^{((d_1,d_2) \in D \oplus D \mapsto (d_1,d_2) \in D^2)}(x) \\ &= \mathcal{A}^{((d_1,d_2) \in D \oplus D \mapsto (d_1,d_2) \in D^2)}(y) \end{aligned} \tag{1}$$

By using the quasi-colimit diagram of small objects at page 92 of Lavendhomme [5], we are sure that there exists a unique $z \in \mathcal{A}_m^{D^2 \oplus D}$ with

$$\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)}(z) = x \tag{2}$$

and

$$\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)}(z) = y \tag{3}$$

We define $y \dot{-} x \in \mathcal{A}_m^1$ to be $\mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)}(z)$.

Proposition 9 *Let $x, y \in \mathcal{A}^2$ abide by (1). Then we have*

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)}(y) \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)}(x) \\
&= y \dot{-} x
\end{aligned}$$

Proof. Let $z \in \mathcal{A}_m^{D^2 \oplus D}$ obedient to (2) and (3). Then we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_2,d_1,d_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1,0)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)}(x)
\end{aligned}$$

while we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_2,d_1,d_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1,d_1d_2)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)}(y)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)}(y) \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_2,d_1)\in D^2)}(x) \\
&= \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_2,d_1,d_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)}(z) \\
&= y \dot{-} x
\end{aligned}$$

This completes the proof. ■

Proposition 10 *Let $x, y \in \mathcal{A}^2$ abide by (1). Then we have*

$$x \dot{-} y = -(y \dot{-} x)$$

Proof. Let $z \in \mathcal{A}^{D^2 \oplus D}$ abide by the conditions (2) and (3). Let $u \in \mathcal{A}^{D^2 \oplus D}$ be

$$u = \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3)\in D^2\oplus D)}(z)$$

Then we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)}(u) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)}(z) \\
&= y
\end{aligned}$$

while we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)}(u) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)}(z) \\
&= x
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&x \dot{-} y \\
&= \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)}(u) \\
&= \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{(d\in D \mapsto (0,0,-d)\in D^2\oplus D)}(z) \\
&= -(y \dot{-} x)
\end{aligned}$$

This completes the proof. ■

Proposition 11 *Let $x, y \in \mathcal{A}^2$ abide by (1) with $a \in \mathbb{R}$. Then we have*

$$a \underset{i}{\cdot} y \dot{-} a \underset{i}{\cdot} x = a(y \dot{-} x) \qquad (i = 1, 2)$$

Proof. Here we deal only with the case $i = 1$, leaving the other case to the reader. Let $z \in \mathcal{A}^{D^2 \oplus D}$ abide by the conditions (2) and (3). Let $u \in \mathcal{A}^{D^2 \oplus D}$ be

$$u = \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (ad_1,d_2,ad_3)\in D^2\oplus D)}(z)$$

Then we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)}(u) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (ad_1,d_2,ad_3)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (ad_1,d_2,0)\in D^2\oplus D)}(z) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (ad_1,d_2)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)}(z) \\
&= a \underset{1}{\cdot} x
\end{aligned}$$

while we have

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(d_1,d_2,d_1d_2)\in D^2\oplus D)}(u)\\
&=\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(d_1,d_2,d_1d_2)\in D^2\oplus D)}\circ\mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D\mapsto(ad_1,d_2,ad_3)\in D^2\oplus D)}(z)\\
&=\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(ad_1,d_2,ad_1d_2)\in D^2\oplus D)}(z)\\
&=\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(ad_1,d_2)\in D^2)}\circ\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(d_1,d_2,d_1d_2)\in D^2\oplus D)}(z)\\
&=a\underset{1}{\cdot}y
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&a\underset{i}{\cdot}y\,\dot{-}\,a\underset{i}{\cdot}x\\
&=\mathcal{A}^{(d\in D\mapsto(0,0,d)\in D^2\oplus D)}(u)\\
&=\mathcal{A}^{(d\in D\mapsto(0,0,d)\in D^2\oplus D)}\circ\mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D\mapsto(ad_1,d_2,ad_3)\in D^2\oplus D)}(z)\\
&=\mathcal{A}^{(d\in D\mapsto(0,0,ad)\in D^2\oplus D)}(z)\\
&=\mathcal{A}^{(d\in D\mapsto ad\in D)}\circ\mathcal{A}^{(d\in D\mapsto(0,0,d)\in D^2\oplus D)}(z)\\
&=a(y\,\dot{-}\,x)
\end{aligned}$$

This completes the proof. ■

Lemma 12 *The following diagram is a quasi-colimit diagram:*

$$\begin{tikzcd}
 & D^2 \arrow[dr, "\varphi_1"] & & D\oplus D \arrow[ll, "\mathbf{i}"'] \arrow[dr, "\mathbf{i}"] & \\
D\oplus D \arrow[ur, "\mathbf{i}"] \arrow[dr, "\mathbf{i}"'] & & D^2\oplus D\oplus D & & D^2 \arrow[ll, "\varphi_3"'] \\
 & D^2 \arrow[ur, "\varphi_2"'] & & D\oplus D \arrow[ll, "\mathbf{i}"] \arrow[ur, "\mathbf{i}"'] &
\end{tikzcd}$$

where $\mathbf{i}:D\oplus D\to D^2$ *is the canonical injection, and* $\varphi_1,\varphi_2,\varphi_3:D^2\to D^2\oplus D\oplus D$ *are defined to be*

$$\begin{aligned}
\varphi_1(d_1,d_2)&=(d_1,d_2,0,0)\\
\varphi_2(d_1,d_2)&=(d_1,d_2,d_1d_2,0)\\
\varphi_3(d_1,d_2)&=(d_1,d_2,0,d_1d_2)
\end{aligned}$$

Proposition 13 *Let* $x,y,z\in\mathcal{A}^2$ *with*

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(x)\\
&=\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(y)\\
&=\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(z)
\end{aligned}$$

Then we have

$$(y\,\dot{-}\,x)+(z\,\dot{-}\,y)+(x\,\dot{-}\,z)=\mathbf{0}$$

Proof. Let $u \in \mathcal{A}^{D^2 \oplus D \oplus D}$ be the unique one such that

$$x = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0,0)\in D^2\oplus D\oplus D)}(u)$$
$$y = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2,0)\in D^2\oplus D\oplus D)}(u)$$
$$z = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0,d_1d_2)\in D^2\oplus D\oplus D)}(u)$$

The unique existence of such $u \in \mathcal{A}^{D^2 \oplus D \oplus D}$ is guaranteed by the above lemma. Since we have

$$\begin{aligned} & x \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0,0)\in D^2\oplus D\oplus D)}(u) \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_3,0)\in D^2\oplus D\oplus D)}(u) \end{aligned}$$

and

$$\begin{aligned} & y \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2,0)\in D^2\oplus D\oplus D)}(u) \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_3,0)\in D^2\oplus D\oplus D)}(u) \end{aligned}$$

we have

$$\begin{aligned} & y \dot{-} x \\ & = \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_3,0)\in D^2\oplus D\oplus D)}(u) \\ & = \mathcal{A}^{(d\in D \mapsto (0,0,d,0)\in D^2\oplus D\oplus D)}(u) \end{aligned} \tag{4}$$

Since we have

$$\begin{aligned} & y \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2,0)\in D^2\oplus D\oplus D)}(u) \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3,d_3)\in D^2\oplus D\oplus D)}(u) \end{aligned}$$

and

$$\begin{aligned} & z \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0,d_1d_2)\in D^2\oplus D\oplus D)}(u) \\ & = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3,d_3)\in D^2\oplus D\oplus D)}(u) \end{aligned}$$

we have

$$\begin{aligned} & z \dot{-} y \\ & = \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,d_1d_2-d_3,d_3)\in D^2\oplus D\oplus D)}(u) \\ & = \mathcal{A}^{(d\in D \mapsto (0,0,-d,d)\in D^2\oplus D\oplus D)}(u) \end{aligned} \tag{5}$$

Since we have

$$
\begin{aligned}
& z \\
& = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0,d_1d_2)\in D^2\oplus D\oplus D)}(u) \\
& = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,0,d_1d_2-d_3)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
$$

and

$$
\begin{aligned}
& x \\
& = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,0,0)\in D^2\oplus D\oplus D)}(u) \\
& = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,0,d_1d_2-d_3)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
$$

we have

$$
\begin{aligned}
& x \dot{-} z \\
& = \mathcal{A}^{(d\in D \mapsto (0,0,d)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D \mapsto (d_1,d_2,0,d_1d_2-d_3)\in D^2\oplus D\oplus D)}(u) \\
& = \mathcal{A}^{(d\in D \mapsto (0,0,0,-d)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
\tag{6}
$$

Since we have

$$
\begin{aligned}
& y \dot{-} x \\
& = \mathcal{A}^{(d\in D \mapsto (0,0,d,0)\in D^2\oplus D\oplus D)}(u) \quad \text{[by (4)]} \\
& = \mathcal{A}^{(d\in D \mapsto (d,0)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (0,0,d_1-d_2,d_2)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
$$

and

$$
\begin{aligned}
& z \dot{-} y \\
& = \mathcal{A}^{(d\in D \mapsto (0,0,-d,d)\in D^2\oplus D\oplus D)}(u) \quad \text{[by (5)]} \\
& = \mathcal{A}^{(d\in D \mapsto (0,d)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (0,0,d_1-d_2,d_2)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
$$

we have

$$
\begin{aligned}
& (y \dot{-} x) + (z \dot{-} y) \\
& = \mathcal{A}^{(d\in D \mapsto (d,d)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (0,0,d_1-d_2,d_2)\in D^2\oplus D\oplus D)}(u) \\
& = \mathcal{A}^{(d\in D \mapsto (0,0,0,d)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
\tag{7}
$$

Since we have

$$
\begin{aligned}
& (y \dot{-} x) + (z \dot{-} y) \\
& = \mathcal{A}^{(d\in D \mapsto (0,0,0,d)\in D^2\oplus D\oplus D)}(u) \quad \text{[by (7)]} \\
& = \mathcal{A}^{(d\in D \mapsto (d,0)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (0,0,0,d_1-d_2)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
$$

and

$$
\begin{aligned}
& x \dot{-} z \\
& = \mathcal{A}^{(d\in D\mapsto(0,0,0,-d)\in D^2\oplus D\oplus D)}(u) \qquad \text{[by (6)]} \\
& = \mathcal{A}^{(d\in D\mapsto(0,d)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(0,0,0,d_1-d_2)\in D^2\oplus D\oplus D)}(u)
\end{aligned}
$$

we have

$$
\begin{aligned}
& \{(y \dot{-} x) + (z \dot{-} y)\} + (x \dot{-} z) \\
& = \mathcal{A}^{(d\in D\mapsto(d,d)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(0,0,0,d_1-d_2)\in D^2\oplus D\oplus D)}(u) \\
& = \mathcal{A}^{(d\in D\mapsto(0,0,0,0)\in D^2\oplus D\oplus D)}(u) \\
& = \mathbf{0}
\end{aligned}
$$

This completes the proof. ■

Let $x, y \in \mathcal{A}_m^3$ with

$$
\begin{aligned}
& \mathcal{A}^{((d_1,d_2,d_3)\in D\times(D\oplus D)\mapsto(d_1,d_2,d_3)\in D^3)}(x) \\
& = \mathcal{A}^{((d_1,d_2,d_3)\in D\times(D\oplus D)\mapsto(d_1,d_2,d_3)\in D^3)}(y)
\end{aligned} \tag{8}
$$

. By using the first quasi-colimit diagram of small objects in Lemma 2.1 of Nishimura [10], we are sure that there exists a unique $z \in \mathcal{A}_m^{D^4\{(2,4),(3,4)\}}$ with

$$
\mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_2,d_3,0)\in D^4\{(2,4),(3,4)\})}(z) = x
$$

and

$$
\mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_2,d_3,d_2d_3)\in D^4\{(2,4),(3,4)\})}(z) = y
$$

We define $y \underset{1}{\dot{-}} x \in \mathcal{A}_m^2$ to be $\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(d_1,0,0,d_2)\in D^4\{(2,4),(3,4)\})}(z)$.

Let $x, y \in \mathcal{A}_m^3$ with

$$
\begin{aligned}
& \mathcal{A}^{((d_1,d_2,d_3)\in D^3\{(1,3)\}\mapsto(d_1,d_2,d_3)\in D^3)}(x) \\
& = \mathcal{A}^{((d_1,d_2,d_3)\in D^3\{(1,3)\}\mapsto(d_1,d_2,d_3)\in D^3)}(y)
\end{aligned} \tag{9}
$$

By using the second quasi-colimit diagram of small objects in Lemma 2.1 of Nishimura [10], we are sure that there exists a unique $z \in \mathcal{A}_m^{D^4\{(1,4),(3,4)\}}$ with

$$
\mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_2,d_3,0)\in D^4\{(1,4),(3,4)\})}(z) = x
$$

and

$$
\mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_2,d_3,d_1d_3)\in D^4\{(1,4),(3,4)\})}(z) = y
$$

We define $y \underset{2}{\dot{-}} x \in \mathcal{A}_m^2$ to be $\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(0,d_1,0,d_2)\in D^4\{(1,4),(3,4)\})}(z)$.

Let $x, y \in \mathcal{A}_m^3$ with

$$\mathcal{A}^{((d_1,d_2,d_3)\in(D\oplus D)\times D\mapsto(d_1,d_2,d_3)\in D^3)}(x) = \mathcal{A}^{((d_1,d_2,d_3)\in(D\oplus D)\times D\mapsto(d_1,d_2,d_3)\in D^3)}(y) \tag{10}$$

By using the third quasi-colimit diagram of small objects in Lemma 2.1 of Nishimura [10], we are sure that there exists a unique $z \in \mathcal{A}_m^{D^4\{(1,4),(2,4)\}}$ with

$$\mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_2,d_3,0)\in D^4\{(1,4),(2,4)\})}(z) = x$$

and

$$\mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_2,d_3,d_1d_2)\in D^4\{(1,4),(2,4)\})}(z) = y$$

We define $y \underset{3}{\dot{-}} x \in \mathcal{A}_m^2$ to be $\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(0,0,d_1,d_2)\in D^4\{(1,4),(2,4)\})}(z)$.

Proposition 14 *Let $x, y \in \mathcal{A}_m^3$.*

1. *If they satisfy (8), then we have*

$$\begin{aligned} & y \underset{1}{\dot{-}} x \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(y) \underset{2}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(x) \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(y) \underset{3}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(x) \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(y) \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(x) \end{aligned}$$

2. *If they satisfy (9), then we have*

$$\begin{aligned} & y \underset{2}{\dot{-}} x \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(y) \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(x) \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(y) \underset{2}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(x) \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(y) \underset{3}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(x) \end{aligned}$$

3. *If they satisfy (10), then we have*

$$\begin{aligned} & y \underset{3}{\dot{-}} x \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(y) \underset{3}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(x) \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(y) \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(x) \\ &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(y) \underset{2}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(x) \end{aligned}$$

Proof. The proof is similar to that in Proposition 9. The details can safely be left to the reader. ■

Now we have

Theorem 15 *The four strong differences* $\overset{\cdot}{-}$, $\underset{1}{\overset{\cdot}{-}}$, $\underset{2}{\overset{\cdot}{-}}$ *and* $\underset{3}{\overset{\cdot}{-}}$ *satisfy the general Jacobi identity. I.e., given* $x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321} \in \mathcal{A}^3$, *as long as the following three expressions are well defined, they sum up only to vanish:*

$$(x_{123} \underset{1}{\overset{\cdot}{-}} x_{132}) \overset{\cdot}{-} (x_{231} \underset{1}{\overset{\cdot}{-}} x_{321})$$

$$(x_{231} \underset{2}{\overset{\cdot}{-}} x_{213}) \overset{\cdot}{-} (x_{312} \underset{2}{\overset{\cdot}{-}} x_{132})$$

$$(x_{312} \underset{3}{\overset{\cdot}{-}} x_{321}) \overset{\cdot}{-} (x_{123} \underset{3}{\overset{\cdot}{-}} x_{213})$$

Proof. The theorem was already proved in case of the standard Nishimura algebroid $\mathcal{S}_M$ in Nishimura's [11], §3. What we should do is only to reformulate the above proof genuinely in terms of diagrams. The details can safely be left to the reader. ■

3.3 *Nishimura algebroids*$_3$

Definition 16 *A Nishimura algebroid*$_2$ $\mathcal{A}$ *over* M *is called a Nishimura algebroid*$_3$ *over* M *providing that it is endowed with a natural transformation* $\mathbf{a}$ *from* $\mathcal{A}$ *to the standard Nishimura algebroid*$_2$ $\mathcal{S}_M$ *to be called the anchor natural transformation.*

Example 17 *The standard Nishimura algebroid*$_2$ $\mathcal{S}_M$ *over* M *is canonically a Nishimura algebroid*$_3$ *over* M *endowed with the identity natural transformation of* $\mathcal{S}_M$.

Example 18 *Let* G *be a groupoid over* M. *Then the Nishimura algebroid*$_2$ $\mathcal{A}G$ *over* M *is a Nishimura algebroid*$_3$ *over* M *endowed with the anchor natural transformation assigning* $\mathbf{a}_G^{\mathcal{D}} : \mathcal{A}^{\mathcal{D}} \to M^{\mathcal{D}}$ *to each object* $\mathcal{D}$ *in* **Simp**.

3.4 *Nishimura Algebroids*$_4$

We denote by $\otimes_{\mathcal{A}}$, or more simply by $\otimes$, the contravariant functor which assigns $\mathcal{D}_1 \otimes \mathcal{D}_2 = \{(\zeta, x) \in (\mathcal{A}^{\mathcal{D}_2})^{\mathcal{D}_1} \times \mathcal{A}^{\mathcal{D}_1} \mid \mathbf{a}(x) = \pi^{\mathcal{D}_1}(\zeta)\}$ to each object $(\mathcal{D}_1, \mathcal{D}_2)$ in **Simp** $\times$ **Simp** and which assigns $f \otimes g = (\zeta \in (\mathcal{A}^{\mathcal{D}_2})^{\mathcal{D}_1} \mapsto \mathcal{A}^f \circ \zeta \circ \mathcal{A}^g \in (\mathcal{A}^{\mathcal{D}'_2})^{\mathcal{D}'_1}, \mathcal{A}^g) : \mathcal{D}_1 \otimes \mathcal{D}_2 \to \mathcal{D}'_1 \otimes \mathcal{D}'_2$ to each morphism $(f, g) : (\mathcal{D}'_1, \mathcal{D}'_2) \to (\mathcal{D}_1, \mathcal{D}_2)$ in **Simp** $\times$ **Simp**, where $(\mathcal{A}^{\mathcal{D}_2})^{\mathcal{D}_1}$ denotes the space of mappings from the infinitesimal space $\mathcal{D}_1$ to $\mathcal{A}^{\mathcal{D}_2}$, and $\pi^{\mathcal{D}_1}(\zeta)$ assigns $\pi(\zeta(d))$ to each $d \in \mathcal{D}_1$. We denote by $\widetilde{\otimes}_{\mathcal{A}}$, or more simply by $\widetilde{\otimes}$, the contravariant functor which assigns $\mathcal{D}_1 \widetilde{\otimes} \mathcal{D}_2 = \mathcal{A}^{\mathcal{D}_1 \times \mathcal{D}_2}$ to each object $(\mathcal{D}_1, \mathcal{D}_2)$ in **Simp** $\times$ **Simp** and which assigns $f \widetilde{\otimes} g = \mathcal{A}^{f \times g} : \mathcal{A}^{\mathcal{D}_1 \times \mathcal{D}_2} \to \mathcal{A}^{\mathcal{D}'_1 \times \mathcal{D}'_2}$ to each morphism $(f, g) : (\mathcal{D}'_1, \mathcal{D}'_2) \to (\mathcal{D}_1, \mathcal{D}_2)$ in **Simp** $\times$ **Simp**.

Definition 19 *A Nishimura algebroid$_3$ $\mathcal{A}$ over M is called a Nishimura algebroid$_4$ over M providing that it is endowed with a natural isomorphism $*_{\mathcal{A}}$ (denoted more simply $*$ unless there is possible confusion) from the contravariant functor $\otimes$ to the contravariant functor $\widetilde{\otimes}$ abiding by the following conditions:*

1. *For any $(\zeta, x) \in \mathcal{D}_1 \otimes \mathcal{D}_2$ with $(\mathcal{D}_1, \mathcal{D}_2)$ in* **Simp** $\times$ **Simp**, *we have*
$$\pi(\zeta * x) = \pi(x)$$
and
$$\mathbf{a}(\zeta * x) = \mathbf{a}^{\mathcal{D}_1}(\zeta)$$
where $\mathbf{a}^{\mathcal{D}_1}(\zeta)$ assigns $\mathbf{a}(\zeta(d_1))(d_2)$ to each $(d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2$.

2. *Let $\mathbf{i}_j : \mathcal{D}_j \to \mathcal{D}_1 \times \mathcal{D}_2$ be the canonical injection with $\mathbf{p}_j : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_j$ the canonical projection ($j = 1, 2$). Then we have*
$$\mathcal{A}^{\mathbf{i}_1}(\zeta * x) = x$$
and
$$\mathcal{A}^{\mathbf{i}_2}(\zeta * x) = \zeta(0_{\mathcal{D}_2})$$
for any $(\zeta, x) \in \mathcal{D}_1 \otimes \mathcal{D}_2$, while we have
$$\mathcal{A}^{\mathbf{p}_1}(y) = (d \in \mathcal{D}_1 \mapsto \mathbf{0}^{\mathcal{D}_2}_{(\mathbf{a}y)(d)}) * y$$
for any $y \in \mathcal{A}^{\mathcal{D}_1}$ and
$$\mathcal{A}^{\mathbf{p}_2}(z) = (d \in \mathcal{D}_1 \mapsto z) * \mathbf{0}^{\mathcal{D}_1}_{\pi(z)}$$
for any $z \in \mathcal{A}^{\mathcal{D}_2}$.

3. *Let $f \in \mathbb{R}^{\mathcal{D}}$. For any $(\zeta, x) \in (\mathcal{D}_1 \times ... \times \mathcal{D}_n) \otimes \mathcal{D}$, we have*
$$\mathcal{A}^{((d,d_1,...,d_n)\in\mathcal{D}\times\mathcal{D}_1\times...\times\mathcal{D}_n\mapsto(d,d_1,...,d_{i-1},f(d)d_i,d_{i+1},...,d_n)\in\mathcal{D}\times\mathcal{D}_1\times...\times\mathcal{D}_n)}(\zeta * x)$$
$$= \{d \in \mathcal{D} \mapsto \mathcal{A}^{((d_1,...,d_n)\in\mathcal{D}_1\times...\times\mathcal{D}_n\mapsto(d_1,...,d_{i-1},f(d)d_i,d_{i+1},...,d_n)\in\mathcal{D}_1\times...\times\mathcal{D}_n)}\zeta(d) \in \mathcal{A}^{\mathcal{D}_1\times...\times\mathcal{D}_n}\} * x$$
($1 \leq i \leq n$)

4. *For any $x \in \mathcal{A}^{\mathcal{D}_1}$, any $\zeta_1 \in (\mathcal{A}^{\mathcal{D}_2})^{\mathcal{D}_1}$ and any $\zeta_2 \in (\mathcal{A}^{\mathcal{D}_3})^{\mathcal{D}_1\times\mathcal{D}_2}$ with $\mathbf{a}(x) = \pi^{\mathcal{D}_1}(\zeta_1)$ and $\mathbf{a}^{\mathcal{D}_1}(\zeta_1) = \pi^{\mathcal{D}_1\times\mathcal{D}_2}(\zeta_2)$, we have*
$$\zeta_2 * (\zeta_1 * x) = (\zeta_2 *^{D} \zeta_1) * x$$
where $\zeta_2 *^{\mathcal{D}_1} \zeta_1 \in (\mathcal{A}^{\mathcal{D}_1\times\mathcal{D}_2})^{\mathcal{D}_1}$ is defined to be
$$(\zeta_2 *^{\mathcal{D}_1} \zeta_1)(d) = \zeta_2(d, \cdot) * \zeta_1(d)$$
for any $d \in \mathcal{D}_1$.

Remark 20 *What we require in our definition of Nishimura algebroid$_4$ over M is that while multiplication seen in groupoids is no longer in view in Nishimura algebroids, the remnants of multiplication and its associativity are to be still in view. Multiplication seems completely lost in the traditional definition of Lie algebroid.*

Example 21 *The standard Nishimura algebroid$_3$ $\mathcal{S}_M$ over M is canonically a Nishimura algebroid$_4$ over M provided that $\zeta *_{\mathcal{S}_M} x \in \mathcal{S}_M^{\mathcal{D}_1 \times \mathcal{D}_2}$ is defined to be*

$$(d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2 \mapsto \zeta(d_1)(d_2) \in M$$

Example 22 *Let G be a groupoid over M. The Nishimura algebroid$_3$ $\mathcal{A}G$ over M is a Nishimura algebroid$_4$ over M provided that $\zeta *_{\mathcal{A}G} x \in (\mathcal{A}G)^{\mathcal{D}_1 \times \mathcal{D}_2}$ is defined to be*

$$(d_1, d_2) \in \mathcal{D}_1 \times \mathcal{D}_2 \mapsto \zeta(d_1)(d_2)x(d_1) \in G$$

Now we give some results holding for any Nishimura algebroid$_4$ $\mathcal{A}$ over M.

Proposition 23 *There is a bijective correspondence between the mappings $\Phi : D \to \mathcal{A}_m^1$ and the elements $x \in \mathcal{A}_m^2$ with $\mathcal{A}^{(d \in D \mapsto (d,0) \in D^2)}(x) = \mathbf{0}_m^D$.*

Proof. This follows simply from the first condition in the definition of Nishimura algebroid$_4$ over M, which claims that the assignment of $\Phi * \mathbf{0}_m^D \in \mathcal{A}_m^2$ to each mapping $\Phi : D \to \mathcal{A}_m^1$ gives such a bijective correspondence. ■

It is easy to see that

Lemma 24 *Let $\mathbf{p}_1 : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{D}_1$ be the canonical projection as in the second condition of Definition 19. Then we have*

$$\mathcal{A}^{\mathbf{p}_1}(\mathbf{0}_m^{\mathcal{D}_1}) = \mathbf{0}_m^{\mathcal{D}_1 \times \mathcal{D}_2}$$

As an easy consequence of the above proposition, we have

Theorem 25 *Given a Nishimura algebroid$_4$ $\mathcal{A}$ over M with $m \in M$, the $\mathbb{R}$-module $\mathcal{A}_m^1$ is Euclidean.*

Proof. We have already proved that $\mathcal{A}_m^1$ is naturally an $\mathbb{R}$-module. Let $\varphi : D \to \mathcal{A}_m^1$ be a mapping. We will consider another mapping $\Phi : D \to \mathcal{A}_m^1$ defined to be

$$\Phi(d) = \varphi(d) - \varphi(0)$$

for any $d \in D$. Let us consider $x = \Phi * \mathbf{0}_m^D \in \mathcal{A}_m^2$. We have $\mathcal{A}^{(d \in D \mapsto (d,0))}(x) = \mathbf{0}_m^D$, while it is easy to see that $\mathcal{A}^{(d \in D \mapsto (0,d))}(x) = \Phi(0) = \mathbf{0}_m^D$. Therefore there is a unique $y \in \mathcal{A}_m^1$ with $\mathcal{A}^{((d_1,d_2) \in D^2 \mapsto d_1 d_2 \in D)}(y) = x$. Let us consider $\mathcal{A}^{((d_1,d_2) \in D^2 \mapsto d_2 \in D)}(y) = y * \mathbf{0}_m^D \in \mathcal{A}_m^2$. Then it is easy to see that

$$\begin{aligned}
&(d \in D \longmapsto dy) * \mathbf{0}_m^D \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \longmapsto (d_1, d_1 d_2) \in D^2)}(\mathcal{A}^{((d_1,d_2) \in D^2 \mapsto d_2 \in D)}(y)) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto d_1 d_2 \in D)}(y) \\
&= x
\end{aligned}$$

Therefore we have $\Phi * \mathbf{0}_m^D = (d \in D \longmapsto dy) * \mathbf{0}_m^D$, which implies that

$$\varphi(d) - \varphi(0) = dy$$

for any $d \in D$. To see the uniqueness of such $y \in \mathcal{A}_m^1$, let us suppose that some $z \in \mathcal{A}_m^1$ satisfies

$$dz = \mathbf{0}_m^D$$

for any $d \in D$. Since $z * \mathbf{0}_m^D = \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto d_2 \in D)}(z)$, we have

$$\begin{aligned}
&(d \in D \to \mathbf{0}_m^D) * \mathbf{0}_m^D \\
&= (d \in D \longmapsto dz) * \mathbf{0}_m^D \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_1,d_1d_2)\in D^2)}(\mathcal{A}^{((d_1,d_2)\in D^2 \mapsto d_2 \in D)}(z)) \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \mapsto d_1d_2 \in D)}(z)
\end{aligned}$$

Since $(d \in D \to \mathbf{0}_m^D) * \mathbf{0}_m^D = \mathbf{0}_m^{D^2}$ by Lemma 24 and the second condition of Definition 19, the desired uniqueness follows from Proposition 1 (§2.2) of Lavendhomme [5]. ■

Now we will discuss the relationship between $*$ and strong differences.

Proposition 26 *1. For any $\zeta_1, \zeta_2 \in (\mathcal{A}^2)^D$ and any $x \in \mathcal{A}^1$ with*

$$\mathbf{a}(x) = \pi^D(\zeta_1) = \pi^D(\zeta_2)$$

and

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (d_1,d_2)\in D^2)}(\zeta_1(d)) \\
&= \mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (d_1,d_2)\in D^2)}(\zeta_2(d))
\end{aligned}$$

for any $d \in D$, we have

$$(\zeta_2 \dot{-} \zeta_1) * x = \zeta_2 * x \underset{1}{\dot{-}} \zeta_1 * x$$

where $\zeta_2 \dot{-} \zeta_1 \in (\mathcal{A}^1)^D$ is defined to be

$$(\zeta_2 \dot{-} \zeta_1)(d) = \zeta_2(d) \dot{-} \zeta_1(d)$$

for any $d \in D$.

2. For any $x, y \in \mathcal{A}^2$ and any $\zeta \in (\mathcal{A}^1)^{D^2 \oplus D}$ with

$$\mathbf{a}(x) = (d_1, d_2) \in D^2 \mapsto \pi(\zeta(d_1, d_2, 0))$$

$$\mathbf{a}(y) = (d_1, d_2) \in D^2 \mapsto \pi(x(d_1, d_2, d_1d_2))$$

and

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (d_1,d_2)\in D^2)}(x) \\
&= \mathcal{A}^{((d_1,d_2)\in D\oplus D \mapsto (d_1,d_2)\in D^2)}(y)
\end{aligned}$$

we have

$$\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto (d_1,d_2)\in D^2)}$$

$$(\{\zeta\circ(d\in D\mapsto (0,0,d)\in D^2\oplus D)\}*(y\overset{\cdot}{-}x))$$

$$=\{\zeta\circ((d_1,d_2)\in D^2\mapsto (d_1,d_2,d_1d_2)\in D^2\oplus D)\}*y\underset{3}{\overset{\cdot}{-}}$$

$$\{\zeta\circ((d_1,d_2)\in D^2\mapsto (d_1,d_2,0)\in D^2\oplus D)\}*x$$

Proof. It suffices to note that given an object $\mathcal{D}$ in **Simp**, the contravariant functor $\widetilde{\otimes}\mathcal{D}$ (resp. $\mathcal{D}\widetilde{\otimes}$) and therefore the functor $\otimes\mathcal{D}$ (resp. $\mathcal{D}\otimes$) map every quasi-colimit diagram of small objects in **Simp** to a limit diagram. Therefore the proof is merely a reformulation of Proposition 2.6 of Nishimura [10]. The details can safely be left to the reader. ■

3.5 *Nishimura Algebroids*$_5$

Definition 27 *A Nishimura algebroid$_4$ $\mathcal{A}$ over M is called a Nishimura algebroid$_5$ over M providing that the anchor natural transformation* $\mathbf{a}$ *from $\mathcal{A}$ to the standard Nishimura algebroid$_4$ $\mathcal{S}_M$ is a homomorphism of Nishimura algebroids$_4$ over M. In other words, a Nishimura algebroid$_4$ $\mathcal{A}$ over M is a Nishimura algebroid$_5$ over M providing that for any $(\zeta,x)\in\mathcal{D}_1\otimes_{\mathcal{A}}\mathcal{D}_2$ with $(\mathcal{D}_1,\mathcal{D}_2)$ in* **Simp** $\times$ **Simp**, *we have*

$$\mathbf{a}(\zeta *_{\mathcal{A}} x)=\mathbf{a}^{\mathcal{D}_1}(\zeta)*_{\mathcal{S}_M}\mathbf{a}(x)$$

Example 28 *It is trivial to see that the standard Nishimura algebroid$_4$ $\mathcal{S}_M$ over M is a Nishimura algebroid$_5$ over M, since* $\mathbf{a}$ *is the identity transformation.*

Example 29 *Let G be a groupoid over M. It is easy to see that the Nishimura algebroid$_4$ $\mathcal{A}G$ over M is a Nishimura algebroid$_5$ over M. It is also easy to see that a homomorphism $\varphi: G\to G'$ of groupoids over M naturally gives rise to a homomorphism $\mathcal{A}\varphi:\mathcal{A}G\to\mathcal{A}G'$ of Nishimura algebroids$_5$ over M. Thus we obtain a functor $\mathcal{A}$ from the category of groupoids over M to the category of Nishimura algebroids$_5$ over M.*

The following proposition should be obvious.

Proposition 30 *Let $\varphi:\mathcal{A}\to\mathcal{A}'$ be a homomorphism of Nishimura algebroids$_5$ over M. Then its kernel at each $m\in M$, denoted by $\ker_m\varphi$, assigning $(\ker_m\varphi)^{\mathcal{D}}=\{x\in\mathcal{A}^{\mathcal{D}}\mid\varphi(x)=0_m^{\mathcal{D}}\}$ to each object $\mathcal{D}$ in* **Simp** *and assigning the restriction $(\ker_m\varphi)^f:(\ker_m\varphi)^{\mathcal{D}'}\to(\ker_m\varphi)^{\mathcal{D}}$ of $\mathcal{A}^f:\mathcal{A}^{\mathcal{D}'}\to\mathcal{A}^{\mathcal{D}}$ to each morphism $f:\mathcal{D}\to\mathcal{D}'$ in* **Simp** *is naturally a Nishimura algebroid$_5$ over a single point.*

3.6 *Nishimura Algebroids*$_6$

Let $\mathcal{A}$ be a Nishimura algebroid$_5$ over M. Since the anchor natural transformation $\mathbf{a}_{\mathcal{A}}:\mathcal{A}\to\mathcal{S}_M$ is really a homomorphism of Nishimura algebroids$_5$ over M,

its kernel $\ker_m \mathbf{a}_{\mathcal{A}}$ at each $m \in M$ is a Nishimura algebroid$_5$ over a single point by dint of the last proposition of the previous subsection. By collecting $\ker_m \mathbf{a}_{\mathcal{A}}$ over all $m \in M$, we obtain a bundle of Nishimura algebroids$_5$ over a single point, which is called the *inner subalgebroid* of $\mathcal{A}$ and which is denoted by $\mathbf{I}\mathcal{A}$. The reader should note that the inner subalgebroid $\mathbf{I}\mathcal{A}$ of $\mathcal{A}$ can naturally be reckoned as a Nishimura algebroid$_5$ over M (as a subalgebroid of $\mathcal{A}$ in a natural sense). In the next definition we will consider the frame groupoid of Nishimura algebroids$_5$ over a single point for $\mathbf{I}\mathcal{A}$, which is denoted by $\Phi_{Nishi_5}(\mathbf{I}\mathcal{A})$.

Definition 31 *A Nishimura algebroid$_5$ $\mathcal{A}$ over M is called a Nishimura algebroid$_6$ over M providing that it is endowed with a homomorphism $\mathrm{ad}_{\mathcal{A}}$ (usually written simply ad) of Nishimura algebroids$_5$ over M from $\mathcal{A}$ to $\mathcal{A}(\Phi_{Nishi_5}(\mathbf{I}\mathcal{A}))$ abiding by the following condition:*

1. *We have*

$$\mathrm{ad}(x)(d_1) \circ \mathrm{ad}(y)(d_2) = (\mathrm{ad}((\mathrm{ad}(x)(d_1))(y)))(d_2) \circ \mathrm{ad}(x)(d_1)$$

   for any objects $\mathcal{D}_1, \mathcal{D}_2$ in **Simp**, *any $d_1 \in \mathcal{D}_1$, any $d_2 \in \mathcal{D}_2$, any $x \in \mathcal{A}^{\mathcal{D}_1}$ and any $y \in (\mathbf{I}\mathcal{A})^{\mathcal{D}_2}$ with $\pi(x) = \pi(y)$.*

2. *Given $x, y \in (\mathbf{I}\mathcal{A})^1$ with $\pi(x) = \pi(y)$, we have*

$$(\mathrm{ad}(x))(d)(y) - y = d[x, y]$$

   for any $d \in D$.

Example 32 *Since the inner subalgebroid $\mathbf{I}\mathcal{S}_M$ of the standard Nishimura algebroid$_5$ $\mathcal{S}_M$ is trivial, $\mathcal{S}_M$ is trivially a Nishimura algebroid$_6$ over M.*

Example 33 *Let G be a groupoid over M. By assigning a mapping*

$$y \in (\mathbf{I}G)_{\alpha x} \mapsto xyx^{-1} \in (\mathbf{I}G)_{\beta x}$$

to each $x \in G$, we get a homomorphism of groupoids over M from G to $\Phi_{grp}(\mathbf{I}G)$, which naturally gives rise to a homomorphism of groupoids over M from G to $\Phi_{Nishi_5}(\mathcal{A}(\mathbf{I}G))$. Since $\mathcal{A}(\mathbf{I}G)$ and $\mathbf{I}(\mathcal{A}G)$ can naturally be identified, we have a homomorphism of groupoids over M from G to $\Phi_{Nishi_5}(\mathbf{I}(\mathcal{A}G))$, to which we apply the functor $\mathcal{A}$ so as to get the desired $\mathrm{ad}_{\mathcal{A}G}$ as a homomorphism of Nishimura algebroids$_5$ over M from $\mathcal{A}G$ to $\mathcal{A}(\Phi_{Nishi_5}(\mathbf{I}(\mathcal{A}G)))$.

4 Totally Intransitive Nishimura Algebroids

Definition 34 *A Nishimura algebroid $\mathcal{A}$ over M is said to be totally intransitive providing that its anchor natural transformation $\mathbf{a}_{\mathcal{A}}$ is trivial, i.e.,*

$$\mathbf{a}_{\mathcal{A}}(x) = \mathbf{0}_m^{\mathcal{D}}$$

for any $m \in M$, any object $\mathcal{D}$ in **Simp** *and any $x \in \mathcal{A}_m^{\mathcal{D}}$.*

Remark 35 *A totally intransitive Nishimura algebroid $\mathcal{A}$ over M can naturally be regarded as a bundle of Nishimura algebroids over a single point over M.*

In this section an arbitrarily chosen totally intransitive Nishimura algebroid $\mathcal{A}$ over M shall be fixed.

Definition 36 *Given $x \in \mathcal{A}^{\mathcal{D}_1}$ and $y \in \mathcal{A}^{\mathcal{D}_2}$ with $\pi(x) = \pi(y)$, we define $x \circledast y \in \mathcal{A}^{\mathcal{D}_1 \times \mathcal{D}_2}$ to be*

$$(d \in \mathcal{D}_2 \mapsto x) * y$$

Proposition 37 *For any $x \in \mathcal{A}^{\mathcal{D}_1}$, $y \in \mathcal{A}^{\mathcal{D}_2}$ and $z \in \mathcal{A}^{\mathcal{D}_3}$ with $\pi(x) = \pi(y) = \pi(z)$, we have*

$$x \circledast (y \circledast z) = (x \circledast y) \circledast z$$

Proof. This follows simply from the fourth condition in Definition 19. ■

Remark 38 *By this proposition we can omit parentheses in a combination by $\circledast$.*

The following proposition is the Nishimura algebroid counterpart of Proposition 3 (§3.2) of Lavendhomme [5].

Proposition 39 *Let $x \in \mathcal{A}^1$. Then we have*

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D(2)\longmapsto d_1+d_2\in D)}(x) \\
&= \mathcal{A}^{((d_1,d_2)\in D(2)\longmapsto (d_1,d_2)\in D^2)}(x \circledast x) \\
&= \mathcal{A}^{((d_1,d_2)\in D(2)\longmapsto (d_2,d_1)\in D^2)}(x \circledast x)
\end{aligned}$$

Proof. Let $z = \mathcal{A}^{((d_1,d_2)\in D(2)\longmapsto (d_1,d_2)\in D^2)}(x \circledast x)$. Then we have

$$\begin{aligned}
&\mathcal{A}^{(d\in D\longmapsto (d,0)\in D(2))}(z) \\
&= \mathcal{A}^{(d\in D\longmapsto (d,0)\in D^2)}(x \circledast x) \\
&= x
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{A}^{(d\in D\longmapsto (0,d)\in D(2))}(z) \\
&= \mathcal{A}^{(d\in D\longmapsto (0,d)\in D^2)}(x \circledast x) \\
&= (d \in D \longmapsto x)(0) \\
&= x
\end{aligned}$$

Therefore the desired first equality follows at once from the quasi-colimit diagram in Proposition 6 (§2.2) of Lavendhomme [5]. The desired second equality can be dealt with similarly. ■

The following proposition is the Nishimura algebroid counterpart of Proposition 6 (§3.2) of Lavendhomme [5].

Proposition 40 *Let $x, y \in \mathcal{A}^1$ with $\pi(x) = \pi(y)$. Then we have*

$$\begin{aligned}&x + y\\&= \mathcal{A}^{(d\in D\longmapsto(d,d)\in D^2)}(y \circledast x)\\&= \mathcal{A}^{(d\in D\longmapsto(d,d)\in D^2)}(x \circledast y)\end{aligned}$$

Proof. Let $z = \mathcal{A}^{(d_1,d_2)\in D(2)\longmapsto(d_1,d_2)\in D^2)}(y \circledast x)$. Then we have

$$\begin{aligned}&\mathcal{A}^{(d\in D\longmapsto(d,0)\in D(2))}(z)\\&= \mathcal{A}^{(d\in D\longmapsto(d,0)\in D^2)}(y \circledast x)\\&= x\end{aligned}$$

and

$$\begin{aligned}&\mathcal{A}^{(d\in D\longmapsto(0,d)\in D(2))}(z)\\&= \mathcal{A}^{(d\in D\longmapsto(0,d)\in D^2)}(y \circledast x)\\&= y\end{aligned}$$

Therefore it follows from the quasi-colimit diagram in Proposition 6 (§ 2.2) of Lavendhomme [5] that

$$\begin{aligned}&x + y\\&= \mathcal{A}^{(d\in D\longmapsto(d,d)\in D(2))}(z)\\&= \mathcal{A}^{(d\in D\longmapsto(d,d)\in D^2)}(y \circledast x)\end{aligned}$$

which establishes the first desired equality. The second desired equality follows similarly. ■

Proposition 41 *Given $x, y \in \mathcal{A}^1$ with $\pi(x) = \pi(y)$, there exists a unique $z \in \mathcal{A}^1$ with $\pi(x) = \pi(y) = \pi(z)$ such that*

$$\begin{aligned}&\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_1d_2\in D)}(z)\\&= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y \circledast x \circledast y \circledast x)\end{aligned}$$

Proof. We will show that

$$\begin{aligned}&\mathcal{A}^{(d\in D\mapsto(d,0)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y \circledast x \circledast y \circledast x)\\&= \mathbf{0}^D_{\pi(x)}\end{aligned}$$

and

$$\begin{aligned}&\mathcal{A}^{(d\in D\mapsto(0,d)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y \circledast x \circledast y \circledast x)\\&= \mathbf{0}^D_{\pi(y)}\end{aligned}$$

Then the desired result will follow from the quasi-colimit diagram in Proposition 7 (§2.2) of Lavendhomme [5]. Now we deal with the first desired identity. Since the composition of $d \in D \mapsto (d,0) \in D^2$ and $(d_1,d_2) \in D^2 \longmapsto (d_1,d_2,-d_1,-d_2) \in D^4$ is equal to the composition of $d \in D \mapsto (d,d) \in D^2$ and $(d_1,d_2) \in D^2 \longmapsto (d_1,0,-d_2,0) \in D^4$, we have

$$\begin{aligned}
&\mathcal{A}^{(d\in D\mapsto(d,0)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y \circledast x \circledast y \circledast x) \\
&= \mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,0,-d_2,0)\in D^4)}(y \circledast x \circledast y \circledast x) \\
&= \mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)}(\mathcal{A}^{(d_2\in D\longmapsto(-d_2,0)\in D^2)}(y \circledast x) \circledast \mathcal{A}^{(d_1\in D\longmapsto(d_1,0)\in D^2)}(y \circledast x)) \\
&= \mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)}((-x) \circledast x) \\
&= x - x \qquad \text{[by Proposition 40]} \\
&= \mathbf{0}^D_{\pi(x)}
\end{aligned}$$

Now we turn to the second desired identity. Since the composition of $d \in D \mapsto (0,d) \in D^2$ and $(d_1,d_2) \in D^2 \longmapsto (d_1,d_2,-d_1,-d_2) \in D^4$ is equal to the composition of $d \in D \mapsto (d,d) \in D^2$ and $(d_1,d_2) \in D^2 \longmapsto (0,d_1,0,-d_2) \in D^4$, we have

$$\begin{aligned}
&\mathcal{A}^{(d\in D\mapsto(0,d)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y \circledast x \circledast y \circledast x) \\
&= \mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(0,d_1,0,-d_2)\in D^4)}(y \circledast x \circledast y \circledast x) \\
&= \mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)}(\mathcal{A}^{(d_2\in D\longmapsto(0,-d_2)\in D^2)}(y \circledast x) \circledast \mathcal{A}^{(d_1\in D\longmapsto(0,d_1)\in D^2)}(y \circledast x)) \\
&= \mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)}((-y) \circledast y) \\
&= y - y \qquad \text{[by Proposition 40]} \\
&= \mathbf{0}^D_{\pi(y)}
\end{aligned}$$

The proof is now complete. ■

Notation 42 *We will denote the above* z *by* $[x,y]$.

Proposition 43 *Given* $x, y \in \mathcal{A}^1$ *with* $\pi(x) = \pi(y)$*, we have*

$$[y,x] = -[x,y]$$

Proof. Let $m=\pi(x)=\pi(y)$. We have

$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_1d_2\in D)}\circ\mathcal{A}^{(d\in D\mapsto(d,d)\in D^2)}([x,y]\circledast[y,x])$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1d_2,d_1d_2)\in D^2)}([x,y]\circledast[y,x])$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1,d_2)\in D^4)}\circ\mathcal{A}^{((d_1,d_2,d_3,d_4)\in D^4\mapsto(d_1d_2,d_3d_4)\in D^2)}([x,y]\circledast[y,x])$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1,d_2)\in D^4)}(\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_1d_2\in D)}([x,y])\circledast\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_1d_2\in D)}([y,x]))$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1,d_2)\in D^4)}(\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_1d_2\in D)}([x,y])\circledast$$
$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}\circ\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_1d_2\in D)}([y,x]))$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1,d_2)\in D^4)}(\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y\circledast x\circledast y\circledast x)\circledast$$
$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}\circ\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(x\circledast y\circledast x\circledast y))$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1,d_2)\in D^4)}(\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(y\circledast x\circledast y\circledast x)\circledast$$
$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2,-d_1)\in D^4)}(x\circledast y\circledast x\circledast y))$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1,d_2)\in D^4)}\circ\mathcal{A}^{((d_1,d_2,d_3,d_4)\in D^4\mapsto(d_2,d_1,-d_2,-d_1,d_3,d_4,-d_3,-d_4)\in D^8)}$$
$$(y\circledast x\circledast y\circledast x\circledast x\circledast y\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2,-d_1,d_1,d_2,-d_1,-d_2)\in D^8)}(y\circledast x\circledast y\circledast x\circledast x\circledast y\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2,d_1,d_2,-d_1,-d_2)\in D^7)}\circ\mathcal{A}^{((d_1,d_2,d_3,d_4,d_5,d_6,d_7)\in D^7\mapsto(d_1,d_2,d_3,-d_4,d_4,d_5,d_6,d_7)\in D^8)}$$
$$(y\circledast x\circledast y\circledast x\circledast x\circledast y\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2,d_1,d_2,-d_1,-d_2)\in D^7)}$$
$$(y\circledast x\circledast y\circledast\mathcal{A}^{(d\in D\mapsto(-d,d)\in D^2)}(x\circledast x)\circledast y\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2,d_2,-d_1,-d_2)\in D^6)}(y\circledast x\circledast y\circledast y\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,d_2,-d_1,-d_2)\in D^5)}\circ\mathcal{A}^{((d_1,d_2,d_3,d_4,d_5)\in D^5\longmapsto(d_1,d_2,-d_3,d_3,d_4,d_5)\in D^6)}$$
$$(y\circledast x\circledast y\circledast y\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,d_2,-d_1,-d_2)\in D^5)}(y\circledast x\circledast\mathcal{A}^{(d\in D\mapsto(-d,d)\in D^2)}(y\circledast y)\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_1,-d_2)\in D^4)}(y\circledast x\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2)\in D^3)}\circ\mathcal{A}^{((d_1,d_2,d_3)\in D^3\longmapsto(d_1,d_2,-d_2,d_3)\in D^4)}(y\circledast x\circledast x\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,-d_2)\in D^3)}(y\circledast\mathcal{A}^{(d\in D\mapsto(d,-d)\in D^2)}(x\circledast x)\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_2)\in D^2)}(y\circledast y)$$
$$=\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto d_2\in D)}\circ\mathcal{A}^{(d\in D\mapsto(d,-d)\in D^2)}(y\circledast y)$$
$$=\mathbf{0}_m^{D^2}$$

■

Proposition 44 *Given $x, y \in \mathcal{A}^1$ with $\pi(x) = \pi(y)$, we have*

$$\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(y \circledast x)$$
$$= \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast y) \tag{11}$$

and

$$[x, y] = y \circledast x \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast y) \tag{12}$$

Proof. Our proof is the proof of Proposition 8 (§3.4) of Lavendhomme [5] in disguise. In order to show the identity (11), it suffices, by dint of the quasi-colimit diagram in Proposition 6 (§2.2) of Lavendhomme [5], to show that

$$\mathcal{A}^{(d\in D\longmapsto(d,0)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(y \circledast x)$$
$$= \mathcal{A}^{(d\in D\longmapsto(d,0)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)} \circ$$
$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast y) \tag{13}$$

and

$$\mathcal{A}^{(d\in D\longmapsto(0,d)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(y \circledast x)$$
$$= \mathcal{A}^{(d\in D\longmapsto(0,d)\in D\oplus D)} \circ \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)} \circ$$
$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast y) \tag{14}$$

Since the composition of $d \in D \longmapsto (d, 0) \in D \oplus D$ and $(d_1, d_2) \in D \oplus D \mapsto (d_1, d_2) \in D^2$ is equal to $d \in D \longmapsto (d, 0) \in D^2$, and since the composition of $d \in D \longmapsto (d, 0) \in D \oplus D$, $(d_1, d_2) \in D \oplus D \mapsto (d_1, d_2) \in D^2$ and $(d_1, d_2) \in D^2 \longmapsto (d_2, d_1) \in D^2$ is equal to $d \in D \longmapsto (0, d) \in D^2$, it is easy to see that both sides of the identity (13) are equal to x by the second condition in Definition 19. The identity (14) can be established similarly. Let

$$z = \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D\longmapsto(d_2,d_3,d_1)\in D^3)}(x \circledast [x, y] \circledast y)$$

Then we have

$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,0)\in D^2\oplus D)}(z)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,0)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D\longmapsto(d_2,d_3,d_1)\in D^3)}(x \circledast [x, y] \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,0,d_1)\in D^3)}(x \circledast [x, y] \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)} \circ \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,0,d_2)\in D^3)}(x \circledast [x, y] \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast \mathcal{A}^{(d\in D\longmapsto(d,0)\in D^2)}([x, y] \circledast y))$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast y)$$

while we have

$$\mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1d_2)\in D^2\oplus D)}(z)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,d_1d_2)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D\longmapsto(d_2,d_3,d_1)\in D^3)}(x \circledast [x,y] \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1d_2,d_1)\in D^3)}(x \circledast [x,y] \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1d_2,d_1)\in D^3)}(x \circledast \mathcal{A}^{(d\in D\mapsto -d\in D)} \circ \mathcal{A}^{(d\in D\mapsto -d\in D)}([x,y]) \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1d_2,d_1)\in D^3)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,-d_2,d_3)\in D^3)}(x \circledast \mathcal{A}^{(d\in D\mapsto -d\in D)}([x,y]) \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_1d_2,d_1)\in D^3)}(x \circledast [y,x] \circledast y)$$

[By Proposition 43]

$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_2,d_1,d_1)\in D^4)} \circ \mathcal{A}^{((d_1,d_2,d_3,d_4)\in D^4\mapsto(d_1,d_2d_3,d_4)\in D^3)}(x \circledast [y,x] \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_2,d_1,d_1)\in D^4)}(x \circledast \mathcal{A}^{((d_1,d_2)\in D^2\mapsto d_1d_2\in D)}([y,x]) \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_2,d_1,d_1)\in D^4)}(x \circledast \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_1,d_2,-d_1,-d_2)\in D^4)}(x \circledast y \circledast x \circledast y) \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_2,d_1,d_1)\in D^4)} \circ \mathcal{A}^{((d_1,d_2,d_3,d_4)\in D^4\mapsto(d_1,d_2,d_3,-d_2,-d_3,d_4)\in D^6})$$
$$(x \circledast x \circledast y \circledast x \circledast y \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,-d_2,d_1,d_2,-d_1,d_1)\in D^6)}(x \circledast x \circledast y \circledast x \circledast y \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,d_2,d_1)\in D^4)} \circ \mathcal{A}^{((d_1,d_2,d_3,d_4)\in D^4\longmapsto(d_1,-d_1,d_2,d_3,-d_4,d_4)\in D^6)}$$
$$(x \circledast x \circledast y \circledast x \circledast y \circledast y)$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,d_2,d_1)\in D^4)}(\mathcal{A}^{(d\in D\mapsto(-d,d)\in D^2)}(x \circledast x) \circledast y \circledast x \circledast$$
$$\mathcal{A}^{(d\in D\mapsto(d,-d)\in D^2)}(y \circledast y))$$
$$= \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1,d_2,d_1)\in D^4)}(\mathbf{0}_m^D \circledast y \circledast x \circledast \mathbf{0}_m^D)$$
$$= y \circledast x$$

Therefore we have

$$\begin{aligned}
& y \circledast x \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2\longmapsto(d_2,d_1)\in D^2)}(x \circledast y) \\
& = \mathcal{A}^{(d\in D\mapsto(0,0,d)\in D^2\oplus D)}(z) \\
& = \mathcal{A}^{(d\in D\mapsto(0,0,d)\in D^2\oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^2\oplus D\longmapsto(d_2,d_3,d_1)\in D^3)}(x \circledast [x,y] \circledast y) \\
& = [x,y]
\end{aligned}$$

This completes the proof. ■

Proposition 45 *1. Given $x \in \mathcal{A}^1$ and $y,z \in \mathcal{A}^2$ with $\pi(x) = \pi(y) = \pi(z)$, if we have*

$$\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(y) = \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(z)$$

then we have

$$\mathcal{A}^{((d_1,d_2,d_3)\in D\times(D\oplus D)\mapsto(d_1,d_2,d_3)\in D^3)}(y \circledast x) = \mathcal{A}^{((d_1,d_2,d_3)\in D\times(D\oplus D)\mapsto(d_1,d_2,d_3)\in D^3)}(z \circledast x)$$

and

$$z \circledast x \underset{1}{\dot{-}} y \circledast x = (z \dot{-} y) \circledast x$$

2. *Given $x, y \in \mathcal{A}^2$ and $z \in \mathcal{A}^1$ with $\pi(x) = \pi(y) = \pi(z)$, if we have*

$$\mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(x) = \mathcal{A}^{((d_1,d_2)\in D\oplus D\mapsto(d_1,d_2)\in D^2)}(y)$$

then we have

$$\mathcal{A}^{((d_1,d_2,d_3)\in(D\oplus D)\times D\mapsto(d_1,d_2,d_3)\in D^3)}(z\circledast x) = \mathcal{A}^{((d_1,d_2,d_3)\in(D\oplus D)\times D\mapsto(d_1,d_2,d_3)\in D^3)}(z\circledast y)$$

and

$$\mathcal{A}^{((d_1,d_2)\in D^2\mapsto(d_2,d_1)\in D^2)}(z \circledast y \underset{3}{\dot{-}} z \circledast x) = z \circledast (y \dot{-} x)$$

Proof. This follows simply from Proposition 26. ■

Proposition 46 *Given $x, y, z \in \mathcal{A}^1$ with $\pi(x) = \pi(y) = \pi(z)$, let it be the case that*

$$\begin{aligned}
u_{123} &= z \circledast y \circledast x \\
u_{132} &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_1,d_3,d_2)\in D^3)}(y \circledast z \circledast x) \\
u_{213} &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_1,d_3)\in D^3)}(z \circledast x \circledast y) \\
u_{231} &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_2,d_3,d_1)\in D^3)}(x \circledast z \circledast y) \\
u_{312} &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_1,d_2)\in D^3)}(y \circledast x \circledast z) \\
u_{321} &= \mathcal{A}^{((d_1,d_2,d_3)\in D^3\mapsto(d_3,d_2,d_1)\in D^3)}(x \circledast y \circledast z)
\end{aligned}$$

Then the right-hands of the following three identities are meaningful, and all the three identities hold:

$$\begin{aligned}
[x,[y,z]] &= (u_{123} \underset{1}{\dot{-}} u_{132}) \dot{-} (u_{231} \underset{1}{\dot{-}} u_{321}) \\
[y,[z,x]] &= (u_{231} \underset{2}{\dot{-}} u_{213}) \dot{-} (u_{312} \underset{2}{\dot{-}} u_{132}) \\
[z,[x,y]] &= (u_{312} \underset{3}{\dot{-}} u_{321}) \dot{-} (u_{123} \underset{3}{\dot{-}} u_{213})
\end{aligned}$$

Proof. Here we deal only with the first identity, leaving the other two identities to the reader. We have

$$
\begin{aligned}
&[x,[y,z]] \\
&= [y,z] \circledast x \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(x \circledast [y,z]) \\
&= \{z \circledast y \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(y \circledast z)\} \circledast x \\
&\dot{-} \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(x \circledast \{z \circledast y \dot{-} \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(y \circledast z)\}) \\
&= \{z \circledast y \circledast x \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(y \circledast z) \circledast x\} \\
&\dot{-} \{x \circledast z \circledast y \underset{3}{\dot{-}} x \circledast \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(y \circledast z)\} \\
&= \{z \circledast y \circledast x \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_1,d_3,d_2)\in D^3)}(y \circledast z \circledast x)\} \\
&\dot{-} \{\mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_3,d_1)\in D^3)}(x \circledast z \circledast y) \\
&\underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_3,d_1)\in D^3)}(x \circledast \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_2,d_1)\in D^2)}(y \circledast z))\} \\
&= \{z \circledast y \circledast x \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_1,d_3,d_2)\in D^3)}(y \circledast z \circledast x)\} \\
&\dot{-} \{\mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_3,d_1)\in D^3)}(x \circledast z \circledast y) \\
&\underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_3,d_1)\in D^3)} \circ \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \longmapsto (d_2,d_1,d_3)\in D^3)}(x \circledast y \circledast z)\} \\
&= \{z \circledast y \circledast x \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_1,d_3,d_2)\in D^3)}(y \circledast z \circledast x)\} \\
&\dot{-} \{\mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_2,d_3,d_1)\in D^3)}(x \circledast z \circledast y) \underset{1}{\dot{-}} \mathcal{A}^{((d_1,d_2,d_3)\in D^3 \mapsto (d_3,d_2,d_1)\in D^3)}(x \circledast y \circledast z)\} \\
&= (u_{123} \underset{1}{\dot{-}} u_{132}) \dot{-} (u_{231} \underset{1}{\dot{-}} u_{321})
\end{aligned}
$$

■

Theorem 47 *Given $m \in M$, the Jacobi identity holds for $\mathcal{A}_m^1$ with respect to the Lie bracket $[\cdot,\cdot]$. I.e., we have*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = \mathbf{0}$$

for any $x, y, z \in \mathcal{A}_m^1$.

5 From Nishimura Algebroids to Lie Algebroids

Let $\mathcal{A}$ be a *Nishimura algebroid over M*. It is very easy to see that

Proposition 48 *By assigning $\Gamma(\mathcal{A})^{\mathcal{D}} = \Gamma(\mathcal{A}^{\mathcal{D}})$ to each object $\mathcal{D}$ in* **Simp** *and assigning $\Gamma(\mathcal{A})^f : \gamma \in \Gamma(\mathcal{A}^{\mathcal{D}'}) \mapsto \mathcal{A}^f \circ \gamma \in \Gamma(\mathcal{A}^{\mathcal{D}})$ to each morphism $f : \mathcal{D} \to \mathcal{D}'$*

in **Simp***, we have a Nishimura algebroid*$_2$ $\Gamma(\mathcal{A})$ *over a single point, where* $\Gamma(\mathcal{A}^{\mathcal{D}})$ *denotes the space of global sections of the bundle* $\mathcal{A}^{\mathcal{D}}$ *over* M*. Endowed with the trivial anchor natural transformation, it is a Nishimura algebroid*$_3$ *over a single point.*

Definition 49 *Given* $X \in \Gamma(\mathcal{A}^{\mathcal{D}_1})$ *and* $Y \in \Gamma(\mathcal{A}^{\mathcal{D}_2})$*, we define* $Y \circledcirc X \in \Gamma(\mathcal{A}^{\mathcal{D}_1 \times \mathcal{D}_2})$ *to be*

$$(Y \circledcirc X)_m = (Y \circ \mathbf{a}(X_m)) * X_m$$

for any $m \in M$*.*

Now we have

Proposition 50 *Given* $X \in \Gamma(\mathcal{A}^{\mathcal{D}_1})$*,* $Y \in \Gamma(\mathcal{A}^{\mathcal{D}_2})$ *and* $Z \in \Gamma(\mathcal{A}^{\mathcal{D}_3})$*, we have*

$$Z \circledcirc (Y \circledcirc X) = (Z \circledcirc Y) \circledcirc X$$

Proof. Let $m \in M$. We have

$$\begin{aligned}
&(Z \circledcirc (Y \circledcirc X))_m \\
&= (Z \circ \mathbf{a}((Y \circledcirc X)_m)) * (Y \circledcirc X)_m \\
&= (Z \circ (\mathbf{a}(Y) \circledcirc \mathbf{a}(X))_m) * \{(Y \circ \mathbf{a}(X_m)) * X_m\} \\
&= [\{m' \in M \mapsto (Z \circ \mathbf{a}(Y_{m'})) * Y_{m'}\} \circ \mathbf{a}(X_m)] * X_m \\
&\text{[By the fourth condition in Definition 19]} \\
&= ((Z \circledcirc Y) \circledcirc X)_m
\end{aligned}$$

■

Remark 51 *By this proposition we can omit parentheses in a combination by* $\circledcirc$*.*

Proposition 52 *By adopting* $\circledcirc$ *as* $*_{\Gamma(\mathcal{A})}$*, our Nishimura algebroid*$_3$ $\Gamma(\mathcal{A})$ *over a single point is a Nishimura algebroid*$_4$ *over a single point.*

Proof. The fourth condition in Definition 19 follows from Proposition 50. The other three conditions follow trivially. ■

Therefore all the discussions of the previous section hold. In particular, we have

Theorem 53 *Given* $X, Y \in \Gamma(\mathcal{A}^1)$*, we can define* $[X, Y] \in \Gamma(\mathcal{A}^1)$ *to be the unique one satisfying*

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto d_1 d_2 \in D)} \circ [X, Y] \\
&= \mathcal{A}^{((d_1,d_2)\in D^2 \longmapsto (d_1,d_2,-d_1,-d_2)\in D^4)} \circ (Y \circledcirc X \circledcirc Y \circledcirc X),
\end{aligned}$$

with respect to which $\Gamma(\mathcal{A}^1)$ *is a Lie algebra.*

Proposition 54 *Given $X, Y \in \Gamma(\mathcal{A}^1)$ and $f \in \mathbb{R}^M$, we have*

$$Y \circledcirc fX = f \underset{1}{\cdot} (Y \circledcirc X)$$

Proof. Let $m \in M$.

$$\begin{aligned}
&(Y \circledcirc fX)_m \\
&= (Y \circ \mathbf{a}(f(m)X_m)) * (f(m)X_m) \\
&= (Y \circ (f(m)\mathbf{a}(X_m))) * (f(m)X_m) \\
&= f(m) \underset{1}{\cdot} (Y \circledcirc X)_m
\end{aligned}$$

■

Proposition 55 *Given $X, Y \in \Gamma(\mathcal{A}^1)$ and $f \in \mathbb{R}^M$, we have*

$$fY \circledcirc X \overset{\cdot}{-} f \underset{2}{\cdot} (Y \circledcirc X) = \mathbf{a}(X)(f)Y$$

Proof. Let $m \in M$. We define $\mu \in \mathcal{A}_m^{D^2 \oplus D}$ to be

$$\mu = \mathcal{A}^{((d_1,d_2,d_3) \in D^2 \oplus D \mapsto (d_1, d_2 f(m) + d_3 \mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m)$$

where $\mathbf{a}(X_m)(f)$ is the Lie derivative of f with respect $\mathbf{a}(X_m)$. It is easy to see that

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1,d_2,d_1 d_2) \in D^2 \oplus D)}(\mu) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1,d_2,d_1 d_2) \in D^2 \oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3) \in D^2 \oplus D \mapsto (d_1, d_2 f(m) + d_3 \mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1, d_2 f(m) + d_1 d_2 \mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1, d_2 f(m) + d_1 d_2 \mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1, d_2 f(\mathbf{a}(X_m)(d_1))) \in D^2)}((Y \circledcirc X)_m) \\
&= (fY \circledcirc X)_m
\end{aligned}$$

[By the third condition in Definition 19]

It is also easy to see that

$$\begin{aligned}
&\mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1,d_2,0) \in D^2 \oplus D)}(\mu) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1,d_2,0) \in D^2 \oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3) \in D^2 \oplus D \mapsto (d_1, d_2 f(m) + d_3 \mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m) \\
&= \mathcal{A}^{((d_1,d_2) \in D^2 \mapsto (d_1, d_2 f(m)) \in D^2)}((Y \circledcirc X)_m) \\
&= (f \underset{2}{\cdot} (Y \circledcirc X))_m
\end{aligned}$$

[By the third condition in Definition 19]

Therefore we have

$$
\begin{aligned}
&(fY \circledcirc X)_m \dot{-} (f \underset{2}{\cdot} (Y \circledcirc X))_m \\
&= \mathcal{A}^{(d \in D \mapsto (0,0,d) \in D^2 \oplus D)}(\mu) \\
&= \mathcal{A}^{(d \in D \mapsto (0,0,d) \in D^2 \oplus D)} \circ \mathcal{A}^{((d_1,d_2,d_3) \in D^2 \oplus D \mapsto (d_1, d_2 f(m) + d_3 \mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m) \\
&= \mathcal{A}^{(d \in D \mapsto (0, d\mathbf{a}(X_m)(f)) \in D^2)}((Y \circledcirc X)_m) \\
&= \mathbf{a}(X_m)(f) Y_m
\end{aligned}
$$

This completes the proof. ■

Proposition 56 *Given $X, Y \in \Gamma(\mathcal{A}^1)$ and $f \in \mathbb{R}^M$, we have*

$$[X, fY] = f[X,Y] + \mathbf{a}(X)(f)Y$$

Proof. We have

$$
\begin{aligned}
&[X, fY] \\
&= fY \circledcirc X \dot{-} \mathcal{A}^{((d_1,d_2) \in D^2 \longmapsto (d_2,d_1) \in D^2)}(X \circledcirc fY) \\
&= \{fY \circledcirc X \dot{-} f \underset{2}{\cdot} (Y \circledcirc X)\} - \\
&\{\mathcal{A}^{((d_1,d_2) \in D^2 \longmapsto (d_2,d_1) \in D^2)}(f \underset{1}{\cdot} (X \circledcirc Y)) \dot{-} f \underset{2}{\cdot} (Y \circledcirc X)\} \\
&[\text{By Proposition 55}] \\
&= \mathbf{a}(X)(f)Y + f[X,Y]
\end{aligned}
$$

This completes the proof. ■

Theorem 57 *Given a Nishimura algebroid $\mathcal{A}$ over M, $\mathcal{A}^1$ is a Lie algebroid over M.*

Proof. This follows directly from Theorem 53 and Proposition 56. ■

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